Lyapunov's indirect method:

- Investigate stability from linearization
- To prove the main result:

① we need to learn how to construct Lyapunov funct. for stable linear sys.

② we need to control the error in linearization.

Review Lyapunov function for linear sys.

- Consider linear system

$$\dot{x} = Ax$$

- Solution is given by

$$x(t) = e^{tA} x(0)$$

- $x = 0$ is GAS iff $Re(\lambda_i) < 0$ for all eigenvalues of A. (Thm 4.5)

- Such a A matrix is called "Hurwitz

- Stability can be studied using Lyapunov function method.

Example:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - 2x_2 \rightsquigarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \Rightarrow \det\left(\begin{bmatrix} \lambda & 1 \\ -1 & \lambda+2 \end{bmatrix}\right) = 0$$

$$\Rightarrow \lambda(\lambda+2) + 1 = 0 \Rightarrow \lambda^2 + 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1$$

$$\Rightarrow A \text{ is Hurwitz} \Rightarrow \text{GAS}$$

- Now we want to show GAS with Lyapunov func.
- Consider quadratic Lyapunov funct

$$V(x) = [x_1 \quad x_2] \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_{P} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

V is p.d if P is positive definite matrix

principal minor condition:

$$p_{11} > 0$$
$$p_{11}p_{22} - p_{12}^2 > 0$$

$$\frac{\partial V}{\partial x}(x) = \left[\frac{\partial V}{\partial x_1}(x), \frac{\partial V}{\partial x_2}(x)\right]$$

$$= \left[2P_{11}x_1 + 2P_{12}x_2, 2P_{22}x_2 + 2P_{12}x_1\right]$$

$$= [x_1, x_2]\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

In general, for $V(x) = x^TPx$, $\frac{\partial V}{\partial x}(x) = 2x^TP$

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)\, f(x) = 2x^TPAx$$

$$= x^T(\underbrace{PA + A^TP}_{-Q})x$$

In general, for any matrix M

$$x^TMx = \frac{1}{2}x^T(\underbrace{M+M^T}_{\text{symmetric part of } M})x$$

why?

$$x^TMx = (x^TMx)^T = x^TM^Tx$$

$$\Rightarrow 2x^TMx = x^TM^Tx + x^TMx = x^T(M+M^T)x$$

$$\Rightarrow x^TMx = \frac{1}{2}x^T(M+M^T)x$$

The Lyapunov funct works if

$$\dot{V}(x) = -x^T Q x < 0 \quad \forall x \neq 0$$

or if Q is positive definite.

$$Q = -PA - A^T P$$

- we choose a positive definite Q and try to find P that gives the Q, or find P that solves

$$\boxed{PA + A^T P = -Q} \rightarrow \text{Lyapunov equation.}$$

- let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

- the LHS is

$$PA + A^T P = \begin{bmatrix} -2P_{12}, & P_{11} - 2P_{12} - P_{22} \\ P_{11} - 2P_{12} - P_{22}, & 2P_{12} - 4P_{22} \end{bmatrix}$$

$$= -Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow P_{12} = \frac{1}{2}, \quad P_{22} = \frac{1}{2}, \quad P_{11} = \frac{3}{2}$$

Principal minor : $P_{11} > 0$ ✓ $\quad P_{11}P_{22} - P_{12}^2 = \frac{1}{2} > 0$ ✓

$\Rightarrow$ $P = \begin{bmatrix} 3/2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ solves $A^T P + PA = -Q$

for $A = \begin{bmatrix} 0 & +1 \\ -1 & -2 \end{bmatrix}$

and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

How to do this in general?

Thm: (thm 4.6 in book)

- If A is Hurwitz, then for all p.d. matrix Q, there is a p.d. matrix P that solves the Lyapunov eq.

$$PA + A^T P = -Q$$

The solution is

$$P = \int_0^\infty e^{tA^T} Q e^{tA} dt$$

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why is this P a solution?

$$P = \int_0^\infty e^{tA^T} Q e^{tA} dt$$

- Because A is Hurwitz, the integral is bounded and P is well-defined

- P is p.d. because $\forall x \neq 0$

$$x^T P x = \int_0^\infty x^T e^{tA^T} Q e^{tA} x \, dt$$

$$= \int_0^\infty \underbrace{(e^{tA} x)^T Q (e^{tA} x)}_{y^T Q y \text{ where } y \neq 0} dt$$

> 0 because Q is p.d.

- To show P solves the Lyapunov eq. consider

$$S(t) = e^{tA^T} Q e^{tA}$$

$$\Rightarrow \dot{S}(t) = A^T S(t) + S(t) A$$

$$\Rightarrow \int_0^\infty \dot{S}(t) dt = A^T \underbrace{\int_0^\infty S(t) dt}_{P} + \underbrace{\int_0^\infty S(t) dt}_{P} A$$

$$\Rightarrow \quad S(\infty) - S(0) = A^T P + PA$$

but $S(\infty) = \lim_{t\to\infty} e^{tA^T} Q e^{tA} = 0$ because A is Hurwitz

and $S(0) = Q$

$$\Rightarrow \quad -Q = A^T P + PA \quad \checkmark$$

<u>Thm</u>: (Thm 4.5 and 4.6 in Khalil)

The following statements are equivalent.

① The system $\dot{x} = Ax$ is AS

② matrix A is Hurwitz

③ $\forall$ p.d. matrix Q, $\exists$ p.d. matrix P that solves

$$PA + A^T P = -Q$$

Thm: (thm 4.7 in Khalil)

Let $x=0$ be eqlb. point of $\dot{x}=f(x)$ where f is C^1. Let

$$A=\frac{\partial f}{\partial x}(0)$$

① If $Re(\lambda_i)<0$ for all eigenvalues of A

$\Rightarrow$ $x=0$ is AS

② If $Re(\lambda_i)>0$ for some eigenvalue of A

$\Rightarrow$ $x=0$ is unstable.

③ If $Re(\lambda_i)\leq 0$ and $Re(\lambda_i)=0$ for some λ_i

$\Rightarrow$ No conclusion

Example:

① $\dot{x} = -x - x^3 \xrightarrow{\text{linearize}} \dot{x} = -x \Rightarrow AS$

② $\dot{x} = -x + x^3 \xrightarrow{\text{"}} \dot{x} = -x \Rightarrow AS$

① is actually GAS, but thm is not strong enough to conclude GAS.

Using the Lyapunov funct.

① $\underbrace{V(x) = x^2}_{\text{p.d and radially unbounded}} \Rightarrow \dot{V}(x) = 2x(-x-x^3)$

$= -2x^2 - 2x^4 < 0 \quad \forall x \neq 0$

$\Rightarrow GAS$

② $V(x) = x^2 \Rightarrow \dot{V}(x) = 2x(-x+x^3)$

$= -2x^2 + 2x^4$

$= -2x^2(1-x^2) < 0$

$\forall x \in D - \{0\}$

where $\underbrace{D = (-1, 1)}_{\text{open set containing 0}} \Rightarrow AS.$